We see that

$$(\text{inner anodyne}) \boxtimes (\text{monos})$$

$$= \overline{(\Lambda^n_k \hookrightarrow \Delta^n \mid 0<k<n)} \boxtimes \overline{(\partial\Delta^n \hookrightarrow \Delta^n)}$$

Lemma 20

$$\subseteq \overline{(\Lambda^n_k \hookrightarrow \Delta^n \mid 0<k<n) \boxtimes (\partial\Delta^n \hookrightarrow \Delta^n)}$$

Lemma 21

$$\subseteq (\text{inner anodyne})$$

$\square$

Now the hard work is done, and we get:

Prop 22: Let $f: X \to Y$ be an (inner, left, right) fibration and $i: A \to B$ a monomorphism. Then

1) $\langle f, i\rangle$ is an (inner, left, right) fibration.

2) If i is moreover (inner, left, right) anodyne, then $\langle f, i\rangle$ is a trivial fibration.

proof: This follows directly from Lemma 18.b) and Prop 19. $\square$

There are many interesting special cases. To keep things simple, we only state the ones for inner anodyne / inner fibrations.

cor 23:

a) if $i : A \to B$ is inner anodyne, so is $i \times id : A \times K \to B \times K$

b) if $p : X \to Y$ is an inner fibration, so if $p_* : Fun(Z, X) \to Fun(Z, Y)$

c) if $i : A \to B$ is a mono and C is an ∞-category, then
$$i^* : Fun(B, C) \to Fun(A, C)$$
is an inner fibration.

d) If $j : A \to B$ is inner anodyne and

C is an ∞-category, then

$$j^* : \mathrm{Fun}(B, C) \longrightarrow \mathrm{Fun}(A, C)$$

is a trivial fibration.

e) If $\begin{cases} K \text{ is any simplicial set} \\ C \text{ is an } \infty\text{-category} \end{cases}$

then $\mathrm{Fun}(K, C)$ is an ∞-category.

proof: They are all obtained by applying Prop 22 to special cases with $A = \emptyset$ or $Y = \Delta^0$. For instance case b) follows from 22. a) with $A = \emptyset$. and case e) follows from b) with $Y = \Delta^0$. $\square$

Here is a nice application: the definition of "composition functors" for ∞-categories which refines Corollary 16.

def 24: Let C be an ∞-category.

The morphism

$$\mathrm{Fun}(\Delta^2, C) \longrightarrow \mathrm{Fun}(I^2, C)$$

induced by $I^2 \subseteq \Delta^2$ is a trivial fibration by Cor. 23 d), hence it admits a section by Cor. 10.2):

$$\sigma : \mathrm{Fun}(I^2, C) \longrightarrow \mathrm{Fun}(\Delta^2, C)$$

By composition, we get

$$\mathrm{Fun}(I^2, C) \xrightarrow{\sigma} \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\underset{\Delta^1}{\overset{}{\Delta^{\{0,2\}}}}, C)$$

where $\Delta^{\{0,2\}} \cong \Delta^1$.

Let us write $\mathcal{O}(C) := \mathrm{Fun}(\Delta^1, C)$

(Lurie's notation; slightly strange, better get used to it)

$\mathcal{O}(C) \underset{t}{\overset{s}{\rightrightarrows}} C$ source / target map.

The above can be written as

$$\mathcal{O}(C) {}_t\!\times_C{}_s \, \mathcal{O}(C) \longrightarrow \mathcal{O}(C)$$

$$I^n = \Delta^1 \amalg_{\Delta^0} \Delta^1 \amalg_{\Delta^0} \cdots$$

This is a composition functor for C.

Similarly, using that $I^n \subseteq \Delta^n$ is inner anodyne (Lemma 15. b)), one can construct composition functors:

$$\mathcal{O}(C) \underset{t\ C\ s}{\times} \mathcal{O}(C) \times \cdots \underset{C}{\times} \mathcal{O}(C) \longrightarrow \mathcal{O}(C). \quad \square$$

Rmk: • For $x \xrightarrow{f} y \xrightarrow{g} z$ in C, write

$$\begin{tikzcd} \mathrm{Comp}(g,f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^2, C) \arrow[d] \\ \Delta^0 \arrow[r, "(f{,}g)"] & \mathrm{Fun}(I^2, C) \end{tikzcd}$$

"the space of composites of f and g".

Then $\mathrm{Comp}(g,f)$ is a fiber of a trivial fibration, hence a contractible Kan complex.

$\Rightarrow$ "the composition is well-defined up to a contractible choice".

($\Leftarrow$ 1- cat. notion of "unique up to a unique isomorphism")

• A related observation: the construction requires choosing a section. However changing the section results in a naturally isomorphic functor, because, as we will see, trivial fibrations are always categorical equivalences.

- The details of the proof of Prop 19 show a slightly more precise statement which is sometimes useful:

Prop 25: We have

$$(\text{inner anodyne}) = \overline{\{\Lambda^2_1 \subseteq \Delta^2\} \boxtimes \{\partial\Delta^n \hookrightarrow \Delta^n\}}$$

$\square$

Cor 26: A simplicial set X is an ∞-category if and only if

$$\mathrm{Fun}(\Delta^2, X) \longrightarrow \mathrm{Fun}(\Lambda^2_1, X)$$

is a trivial fibration.

proof: Write $\mathrm{Fun}(\Delta^2, X) \xrightarrow{g} \mathrm{Fun}(\Lambda^2_1, X)$

We have $g = \langle X \to \Delta^0, \Lambda^2_1 \to \Delta^2 \rangle$.

g trivial fibration.

$\iff (\partial\Delta^n \hookrightarrow \Delta^n) \boxslash g$

Lemma 18.b)

$\iff ((\partial\Delta^n \hookrightarrow \Delta^n) \boxtimes (\Lambda^2_1 \to \Delta^2)) \boxslash (X \to \Delta^0)$

left comp. are saturated

$\iff \overline{(\partial\Delta^n \hookrightarrow \Delta^n) \boxtimes (\Lambda^2_1 \to \Delta^2)} \boxslash (X \to \Delta^0)$

Prop 25.

$\iff$ (inner anodyne) $\boxslash (X \to \Delta^0)$

$\iff X$ ∞-category. $\square$

By the same argument one proves:

Cor 27: A morphism $g: X \to Y$ is an inner fibration iff

$$\mathrm{Fun}(\Delta^2, X) \longrightarrow \mathrm{Fun}(\Delta^2, Y) \times_{\mathrm{Fun}(I^2, Y)} \mathrm{Fun}(I^2, X)$$

is a trivial fibration. $\square$

3) Categorical equivalences

Recall the following definition.

def 27: Let C, D be ∞-categories.
A functor $F : C \to D$ is an equivalence of ∞-categories iff there exists $G : D \to C$ such that
$$\begin{cases} F \circ G \simeq id_D & \text{in } Fun(D,D) \\ G \circ C \simeq id_C & \text{in } Fun(C,C). \end{cases}$$

G is called a categorical inverse of F. $\square$

To study this notion, it is convenient to have better functoriality for $Fun(-,-)$.

Lemma 28: Let X, Y, Z be simplicial sets.

a) There is a morphism

$$comp : Fun(X,Y) \times Fun(Y,Z) \longrightarrow Fun(X,Z)$$

which on 0-simplices coincides with

composition in sSet.

b) There is a morphism

$$(\ )^* : \mathrm{Fun}(X,Y) \longrightarrow \mathrm{Fun}\big(\mathrm{Fun}(Y,Z), \mathrm{Fun}(X,Z)\big)$$

which on 0-simplices coincide with precomposition $F \mapsto (F^* : G \mapsto G \circ F)$

proof:

* Recall the adjunction $(A \times -) \dashv \mathrm{Fun}(A, -)$.

The counit gives us maps

$$\begin{cases} ev_{X,Y} : X \times \mathrm{Fun}(X,Y) \longrightarrow Y \\ ev_{Y,Z} : Y \times \mathrm{Fun}(Y,Z) \longrightarrow Z \end{cases}$$

and so a map

$$ev_{Y,Z} \circ (ev_{X,Y} \times id) : X \times \mathrm{Fun}(X,Y) \times \mathrm{Fun}(Y,Z) \to Z$$

We now define comp by adjunction.

On 0-simplices, we have

$$\mathrm{comp}(F,G)(\underset{\substack{\cap \\ X_n}}{x}) = ev_{Y,Z} \circ (ev_{X,Y} \times id)(x,F,G)$$

$$= ev_{Y,Z}(F(x), G)$$

$$= G(F(x))$$

as we wanted.

b) We simply define $(\ )^*$ from comp by adjunction. $\square$

Cor 29: Let C, D, E be ∞-categories.

The functor comp induces a functor

$$R\,\mathrm{comp} : R\,\mathrm{Fun}(C,D) \times R\,\mathrm{Fun}(D,E) \to R\,\mathrm{Fun}(C,E).$$

proof: This follows from the previous lemma together with $R(C \times D) \simeq RC \times RD$ for ∞-categories. $\square$

def 30: . Let C be a 1-category
The **core** $\mathrm{Core}(C)$ of C is the subcategory of C whose objects are all of $\mathrm{Ob}(C)$ and morphisms are the isomorphisms in C.

- Let C be an ∞-category. The **core** $\mathrm{Core}(C)$ is the subcategory of C corresponding to $\mathrm{Core}(hC) \subseteq hC$.

In other words, there is a pullback square

$$\begin{tikzcd}
\mathrm{Core}(C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
N\,\mathrm{Core}(hC) \arrow[r] & N\,hC
\end{tikzcd}$$

Lemma 31: Let C be an ∞-category. $\mathrm{Core}(C)$ is an ∞-groupoid, and is the largest subcategory of C which is an ∞-groupoid.

proof: exercise. $\square$

Rmk: By construction, we have

$$h(\mathrm{Core}(C)) \simeq \mathrm{Core}(hC)$$

is a groupoid, and so we can define its π_0 as a groupoid. Also by construction, $\pi_0(\mathrm{Core}(hC))$ is the set of isomorphism classes of objects in C.

def 32: Let $\begin{cases} K \in s\mathrm{Set} \\ C \in \mathrm{Cat}_\infty^1 \end{cases}$. We put

$$\mathrm{Map}(K, C) := \mathrm{Core}(\mathrm{Fun}(K, C)).$$

In particular, $\mathrm{Map}(K, C)$ is an ∞-groupoid whose objects are $\{K \to C\}$ and morphisms are natural isomorphisms.

def 33: The homotopy category of ∞-categories $\mathfrak{h}\mathrm{Cat}_\infty$ is a 1-category defined as follows.

The objects are ∞-categories.

The morphisms from C, D are defined by $\pi_0\, \mathfrak{h}\,\mathrm{Map}(C, D)$.

For $F: C \to D$, we write $[F]$ for the class in $\mathfrak{h}\mathrm{Cat}_\infty(C, D)$.

Composition is defined via Corollary 29.

Associativity and unitality follows from similar properties for the map comp, which are left as exercises.

Lemma 34: Let C, D be ∞-categories.

a) A functor $F: C \to D$ is an equivalence iff $[F]$ is an isomorphism in $\mathcal{R}Cat_\infty$.

b) Let $F: C \to D$ be an equivalence of ∞-categories, and $G, G': D \to C$ be two categorical inverses of F. Then G and G' are naturally isomorphic.

proof:

a) This is just a restatement of the definition.

b) By assumption and part a), we have

$$[F] \circ [G] = id_D, \quad [G'] \circ [F] = id_C$$

so

$$\begin{aligned} [G'] &= [G'] \circ ([F] \circ [G]) \\ &= ([G'] \circ [F]) \circ [G] \\ &= [G] \end{aligned}$$

$\square$

We now extend this notion to more general simplicial sets.

def 35: A morphism $F : X \to Y$ in sSet is a *categorical equivalence* if for every ∞-category C, the induced functor

$$\mathrm{Fun}(Y, C) \xrightarrow{F^*} \mathrm{Fun}(X, C)$$

is an equivalence of ∞-categories.

. The following shows that the two notions are compatible.

prop 36: Let $F : C \to D$ be a functor between ∞-categories. TFAE:

1) F is an equivalence.

2) For every ∞-cate ory E, the functor

$$\mathrm{Fun}(D, E) \longrightarrow \mathrm{Fun}(C, E)$$ is an equivalence.

proof:

1) $\Rightarrow$ 2)

* Let's use $(\ )^*$. By assumption, F an equivalence of ∞-categories, so there is $G : D \to C$ such that

$$\begin{cases} F \circ G \simeq id_D & \text{in } Fun(D,D) \\ G \circ F \simeq id_C & \text{in } Fun(C,C) \end{cases}$$

i.e there exist 4 natural transformations

$$\begin{cases} \alpha, \alpha' \in Fun(D,D)_1 \\ \beta, \beta' \in Fun(C,C)_1 \end{cases}$$

$$\begin{cases} \alpha : f \circ g \to id_D \ , \ \alpha' : id_D \to F \circ G \\ \beta : g \circ f \to id_C \ , \ \beta' : id_C \to G \circ F \end{cases}$$

such that $\begin{cases} \alpha \circ \alpha' = id \\ \alpha' \circ \alpha = id \end{cases}$ in $RFun(D,D)$

$$\begin{cases} \beta \circ \beta' = id \\ \beta' \circ \beta = id \end{cases} \quad \text{in } hFun(C, C).$$

We claim that F^* is an equivalence of ∞-categories, with categorical inverse G^*.

The point is that $\alpha^*, (\alpha')^*, \ldots$ define natural isomorphisms $\begin{cases} F^* \circ G^* \simeq id \\ G^* \circ F^* \simeq id. \end{cases}$

and this follows from applying the functoriality of $(\ )^*$ to all the equations above.

<u>2) => 1):</u>

* We apply 2) to $E = C$.

$$F^* : Fun(D, C) \longrightarrow Fun(C, C).$$

is an equivalence of ∞-categories.

In particular, by Cor. II.16,

$$\mathbb{R}F^* : \mathbb{R}\,\mathrm{Fun}(D,C) \longrightarrow \mathbb{R}\,\mathrm{Fun}(C,C)$$

is an equivalence of categories,

and

$$\mathrm{Core}(\mathbb{R}F^*) : \mathbb{R}\,\mathrm{Map}(D,C) \longrightarrow \mathbb{R}\,\mathrm{Map}(C,C)$$

is an equivalence of groupoids,

and

$$\pi_0\,\mathrm{Core}\,\mathbb{R}F^* : \mathbb{R}\,\mathrm{Cat}_\infty(D,C) \longrightarrow \mathbb{R}\,\mathrm{Cat}_\infty(C,C)$$

is a bijection of sets.

So take a preimage of id_C to get

$$G : D \longrightarrow C$$

with a natural isomorphism $G \circ F \longrightarrow id_C$.

Now taking $E = D$, we see

$$F^*(id_D) = id_D \circ F = F \circ id_C \simeq FGF = F^*(F \circ G)$$

so $id_D \simeq F \circ G$ and we are done. $\square$

$\leadsto F^*: \mathrm{Fun}(D,D) \to \mathrm{Fun}(C,D)$

Rmk: Here is a motivation for this definition from simplicial homotopy theory.

A morphism $F: X \to Y$ in sSet is a weak homotopy equivalence if $|F| : |X| \to |Y|$ is an homotopy equivalence.
(or equivalently a weak h. eq. by Whitehead)

Facts:

- If X, Y are Kan complexes, then F is a weak htpy eq. $\Leftrightarrow$ F is an homotopy equivalence ($\exists G: Y \to X \ldots$) $\left| \begin{array}{l} F \circ G \underset{s.}{\simeq} id \\ G \circ F \underset{s.}{\simeq} id \end{array} \right.$
- If $X \in$ sSet, $K \in$ Kan, then $\mathrm{Map}(X, K)$ is a Kan complex.
  $(= \mathrm{Fun}(X, K))$
- In general, F is a weak homotopy equivalence iff for all K Kan complexes, $\mathrm{Map}(Y, K) \xrightarrow{F^*} \mathrm{Map}(X,K)$ is an htpy eq.

* The condition in Def. 35 can be relaxed as follows.

Prop 37: Let $F : X \to Y$ in sSet. Then F is a categorical equivalence iff for all ∞-categories C, the map

$$[F]^* : \pi_0 \operatorname{Map}(Y, C) \longrightarrow \pi_0 \operatorname{Map}(X, C)$$

is a bijection.

proof: $\Rightarrow$ is clear.

$\Leftarrow$: Let C be an ∞-category. We want to show $F^* : \operatorname{Fun}(Y, C) \to \operatorname{Fun}(X, C)$ is an equivalence of ∞-categories, i.e. that $[F^*]$ is an isomorphism in $h\mathrm{Cat}_\infty$.

By the Yoneda lemma in $h\mathrm{Cat}_\infty$, it suffices to show that for any ∞-category D, the induced map

$$\theta : \mathcal{R}\mathrm{Cat}_\infty(D, \mathrm{Fun}(Y,C)) \longrightarrow \mathcal{R}\mathrm{Cat}_\infty(D, \mathrm{Fun}(X,C))$$

is a bijection. But we have for any $M, N, P \in sSet$, an isomorphism

$$\mathrm{Fun}(M, \mathrm{Fun}(N,P)) \simeq \mathrm{Fun}(N, \mathrm{Fun}(M,P))$$
$$\left(\begin{array}{ccc} \wr\!\!\! & & \wr \\ & \mathrm{Fun}(M \times N, P) & \end{array} \right)$$

and we can identify θ with the map

$$\theta' : \pi_0 \mathrm{Map}(Y, \mathrm{Fun}(D,C)) \to \pi_0 \mathrm{Map}(X, \mathrm{Fun}(D,C))$$

which is a bijection by applying the assumption to $\mathrm{Fun}(D, C)$. $\square$

* We now consider some examples of categorical equivalences.

For the next proposition, we are in a bit of a bind. There is a relatively complicated proof in [RezK, 20.10] which I don't want to reproduce here.

I prefer to give a more natural argument (taken from [Kerodon, 4.2.5.9]) which however relies on a fact which we will only prove later (without circular arguments!), namely

Thm: Let $F, F' : C \to D$ be functors of ∞-categories and $u : F \to F'$ be a natural transformation. Then u is a natural isomorphism <u>iff</u> for all $c \in C_0$, the induced map

$$u_c : F(c) \longrightarrow F'(c)$$

is an isomorphism in D.

Rmk The analogue of this for 1-categories is easy: the u_c^{-1} form the components of an inverse of u. The problem is that inverses are not unique in an ∞-category; the Theorem tells you that in this situation, it is possible to choose them "coherently" to get an inverse of u.

Prop 38: Every trivial fibration is a categorical equivalence.

proof: Let $F: X \to Y$ be a trivial fibration, and C be an ∞-category. By Prop 37, it is enough to show that

$$[F]^*: \pi_0 \operatorname{Map}(Y, C) \xrightarrow{\sim} \pi_0 \operatorname{Map}(X, C).$$

By Cor. 10, F admits a section $S : Y \to X$ and there is a simplicial homotopy

$$\begin{tikzcd} X \times \Delta^1 \arrow[r, "H"] \arrow[dr] & X \arrow[d] \\ & Y \end{tikzcd}$$

between id_X and $S \circ F$.

$*$ We have $F \circ S = \mathrm{id}$, hence $[S]^* \circ [F]^* = \mathrm{id}$.

We will finish the proof by showing $[F]^*[S]^* = \mathrm{id}$.

Let $G : X \to C$; we need to prove that G is isomorphic to GSF. The map $G \circ H$ provides a natural transformation

$$G \circ H : G \circ S \circ F \longrightarrow G$$

and we need to prove it is a natural isomorphism.

By the Thm which we admitted, it suffices to do this after evaluating at each $x \in X_0$. The resulting map is the image of

$$\Delta^1 \simeq \{x\} \times \Delta^1 \hookrightarrow X \times \Delta^1 \xrightarrow{H} X \xrightarrow{G} C.$$

$F^{-1}(y)$

Putting $y = F(x) \in Y_0$, we see that this composition factors through $F^{-1}(y)$ because H is an homotopy from id_X to $s \circ F /y$

But $F^{-1}(y)$ is a fibre of a trivial fibrat°, hence in particular a Kan complex, and we know that Kan complexes are ∞-groupoids, so $\Delta^1 \to F^{-1}(y) \xrightarrow{\sim} C$ is also an iso. $\square$

prop 37: Every inner anodyne morphism is a categorical equivalence.

proof: Let $\begin{cases} i : A \to B \text{ be inner anodyne} \\ C \ \ \infty\text{-category} \end{cases}$

By Cor 23. c), the morphism

$$\mathrm{Fun}(B, C) \xrightarrow{i^*} \mathrm{Fun}(A, C)$$

is a trivial fibration, hence (by Prop 38) a categorical equivalence, hence (by Prop 36) an equivalence of ∞-categories. $\square$

Prop 38: Let X be a simplicial set.

1) There exists an ∞-category C and an inner anodyne map (hence a categorical equivalence by Prop 37) $F: X \to C$.

2) Let $F_1, F_2 : X \to C_1, C_2$ be two categorical equivalences with C_1, C_2 ∞-categories. Then there exists a categorical equivalence $g: C_1 \to C_2$, unique up to a natural iso., such that $g f_1 = f_2$.

(In other words, a simplicial set is categorically equivalent to an ∞-category well-defined up to categorical equivalence).

proof: I only prove part 1) and refer you to [Rezk, 20.16] for part 2).

By the small object argument, we can factor $X \to \Delta^0$ into $X \xrightarrow{j} C \xrightarrow{p} \Delta^0$ with

$$\begin{cases} j \text{ inner anodyne } \overset{\text{Prop 37}}{\Longrightarrow} \text{ categorical eq.} \\ p \text{ inner fibration } \Longleftrightarrow C \ \infty\text{-category.} \end{cases}$$

$\square$